Robust Nonlinear $H_\infty$ FIR Filtering for Time-Varying Systems

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Abstract: This paper investigates the robust nonlinear $H_\infty$ filter with FIR (Finite Impulse Response) structure for nonlinear discrete time-varying uncertain systems represented by the state-space model having parameter uncertainty. Firstly, when there is no parameter uncertainty in the system, the discrete-time nominal nonlinear $H_\infty$ FIR filter is derived by using the equivalence relationship between the FIR filter and the recursive filter, which corresponds to the standard nonlinear $H_\infty$ filter. Secondly, when the system has the parameter uncertainty, the robust nonlinear $H_\infty$ FIR filter is proposed for the discrete-time nonlinear uncertain systems.

Keywords: $H_\infty$ FIR filter, nonlinear filtering, robustness, uncertainty, time-varying system

I. Introduction

Over the past several years, the problem of the nonlinear $H_\infty$ filtering has been studied by a number of authors [4][6][12]. There are two approaches commonly used for providing solutions to nonlinear $H_\infty$ control and filtering problems. One is based on the dissipativity theory and the differential game theory. Another is based on the nonlinear version of the classical Bounded Real Lemma developed by Willems [14] and Hill and Moylan [5]. However, the nonlinear $H_\infty$ filters proposed so far are mainly limited to time-invariant systems. Therefore they can not be applied to general time-varying systems on the infinite horizon since one of two Riccati differential equations required to solve the problem can not be computed on the infinite horizon. To solve this problem, Kwon et al. [10][11] have proposed the robust $H_\infty$ FIR filter for general time-varying system. However, their filter is limited to linear systems, and it is not to be directly applied to nonlinear systems. This paper deals with the issue of the robust nonlinear $H_\infty$ filtering problem for discrete time-varying systems with the parameter uncertainties on the infinite horizon. The basic idea of the current paper is to formulate the robust nonlinear $H_\infty$ filtering problem on the discrete-time moving horizon and to adopt the FIR (Finite Impulse Response) filter structure. FIR filters are widely used in the signal processing area, and they were utilized in the estimation problem as the optimal FIR filters [7][8][9]. Since the optimal FIR filters use the finite observations only over a finite preceding time interval, they can overcome the divergence problem and have the built-in BIBO (Bounded Input/Bounded Output) stability and the robustness to the numerical problems such as coefficient quantization errors and roundoff errors, which are well known properties of the FIR structure in signal processing area. Also note that IIR (Infinite Impulse Response) or recursive filter structure requires the initial conditions on each horizon, which is an impractical assumption, but that FIR filter structure does not requires the initial conditions. The optimal FIR filters are, however, presented so far not in the $H_\infty$ setting but in the minimum variance formulation.

The nonlinear $H_\infty$ filter proposed is to be called hereafter as the robust nonlinear $H_\infty$ FIR filter in the sense that it is a nonlinear $H_\infty$ filter with the FIR structure for uncertain systems. It will be shown that the nonlinear $H_\infty$ FIR filter always has a solution if the standard nonlinear $H_\infty$ filter exists on the finite horizon. Therefore, the derivation for $H_\infty$ FIR filter solution for infinite horizon is very simple and trivial. It is noted that the nonlinear filter proposed works on the time-varying nonlinear systems with time-varying parametric uncertainties, and that this point will be one of the main contributions of the current paper.

For the case when there is no parameter uncertainty in the system, we are concerned with designing a nonlinear $H_\infty$ FIR filter such that the induced $l_1$ operator norm of the mapping from the noise signal to the estimation error is within a specified bound. It is shown that this problem can be solved via one Riccati equation. The design of nonlinear filters which guarantee a prescribed $H_\infty$ performance in the presence of parameter uncertainty are also considered. In this situation, a solution is to be obtained in terms of two Riccati equations.

II. Problem formulation

Consider the uncertain nonlinear time-varying system of the form

$$x_{k+1} = (A + \Delta A_k) x_k + G g(x_k) + B w_k$$

$$y_k = (C + \Delta C_k) x_k + H h(x_k) + D w_k$$

$$z_k = L x_k,$$  

(3)

where $x_k \in \mathbb{R}^n$ is the state vector with the initial state $x_0$ unknown, $w_k \in \mathbb{R}^r$ is a noise signal which belongs to $l_2[0, \infty)$, $y_k \in \mathbb{R}^m$ is the measurement, $z_k \in \mathbb{R}^q$ is a linear combination of state variables to be estimated, $g(\cdot):\mathbb{R}^r \to \mathbb{R}^r$ and $h(\cdot):\mathbb{R}^r \to \mathbb{R}^m$ are known nonlinear vector functions and $A, B, C, D, G, H$ and $L$ are known real time-varying matrices of appropriate dimensions that describe the nominal system together with $g(\cdot)$ and $h(\cdot)$. The matrices $\Delta A_k$ and $\Delta C_k$ represent time-varying parameter uncertainties in $A$ and $C$, respectively. These uncertainties are assumed to be of the following structure

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E,$$

(4)

where $F_k$ is an unknown real time-varying matrix satisfying

$$F_k^T F_k \leq I, k = 0,1,2, \cdots$$

(5)
and $H_1$, $H_2$ and $E$ are known real constant matrices of appropriate dimensions that specify how the elements of the nominal matrices $A$ and $C$ are affected by the uncertain parameters in $F_k$.

Assumption 1:
(a) $[D \ H_2 \ H]$ is of the full row rank;
(b) $DB^T = 0$;
(c) $g(0) = 0$;
(d) There exist known constant matrices $V_1$ and $V_2$ such that for any $x_1$ and $x_2 \in \mathbb{R}^n$,

$$
\| g(x_1) - g(x_2) \| \leq \| V_1 (x_1 - x_2) \|
$$

$$
\| h(x_1) - h(x_2) \| \leq \| V_2 (x_1 - x_2) \|
$$

Assumption 1(a) and 1(b) means that the robust $H_\infty$ FIR filtering problem is ‘nonsingular’. Observe that if the parameter uncertainty and the nonlinearity in the output matrix disappear, i.e. $H_2 = 0$ and $H = 0$, Assumption 1(a) reduces to $DD^T > 0$, which is a standard assumption in the $H_\infty$ FIR filtering problem for the nominal system. Assumption 1(c) means that the initial condition of the nonlinearity function in the state matrix is zero.

Observe that discrete-time nonlinear models of the form (1)-(2) can be used to represent many important physical systems. The parameter uncertainty in the linear terms can be regarded as the variation of the operating points of the nonlinear system. We also note that the parameter uncertainty structure of (4) has been widely used in the problems of robust control and robust filtering and can capture the uncertainty in a number of practical situations.

In this section we are concerned with designing a nonlinear causal filter $\mathfrak{F}$ with FIR structure for estimating $z_k$ with a guaranteed performance in a $H_\infty$ sense, using the measurements $Y_{k,i} = \{y_j, j = 0, 1, \ldots, k-1\}$ and where no a priori estimate of the initial state of (1) is assumed. Letting $z_k$ denote the estimate of $z_k$, the filter is required to guarantee a uniformly small estimation error $z_k - z_k$, for any $w \in l_2([0,\infty))$ and $x_k \in \mathbb{R}^n$. Then the robust nonlinear $H_\infty$ FIR filtering problem is formulated as follows:

Given the system (1)-(3) and a prescribed level of noise attenuation $\gamma > 0$ on each horizon $[k-N, k]$, find a causal filter $\mathfrak{F}$ such that the filtering error dynamics is globally uniformly asymptotically stable and $\| z - z_k \| < \gamma \| w \|_{l_2} + x_k^T R x_k$ for any non-zero $w \in \mathbb{R}^n \oplus l_2([0,\infty))$ and for all uncertainties satisfying (4)-(5), where $R > R^T > 0$ is a given weighting matrix for $x_0$ and $\| \cdot \|$ means the direct sum of the vector subspaces. Here, $\| \cdot \|$ denotes $\ell^2$-norm on the infinite horizon and $\| \cdot \|_{l_2}$ denotes the usual $l_2$-norm on the moving horizon $[k-N, k]$.

Provided that there is no parameter uncertainty in the system, i.e., $\Delta A_k = 0$ and $\Delta C_k = 0$ for all $k$ in the above formulation, the problem reduces to the nominal nonlinear $H_\infty$ FIR filtering problem, which corresponds to the nonlinear $H_\infty$ filtering problem.

Note that the performance index in the above problem statements is a worst-case performance measure and can be viewed as a generalization of the standard $H_\infty$ performance measure to deal with unknown initial state. The weighting matrix, $R$, is a measure of the uncertainty in $w$ relative to the uncertainty in $w$. A ‘large’ value of $R$ indicates that the initial state is likely to be very close to zero.

In the current paper, the FIR filter is defined as follows

$$
\hat{z}(k + 1) = \int_{k-N}^{k} T(i;N)y(i)
$$

where $T(k;N)$ is the finite impulse response with the finite duration $N$. This FIR filter is a kind of the one-step-ahead predictor since it estimates the state or the output at the time point $k+1$ based on the observation on $[k-N, k]$. The $H_\infty$ FIR filter is obtained by constructing its impulse response from that of the $H_\infty$ filter on the finite moving horizon $[k-N, k]$.

We end this section by recalling a version of the bounded real lemma for linear discrete time-varying systems, which will be used in the derivation of a solution to the above filtering problems.

Consider the following linear time-varying system

$$
x_{k+1} = A_k x_k + B_k w_k
$$

$$
z_k = C_k x_k
$$

where $x_k \in \mathbb{R}^n$ is the state vector with the initial state $x_0$ being unknown, $w_k \in \mathbb{R}^n$ is the input which belongs to $l_2([0,\infty))$, $z_k \in \mathbb{R}^p$ is the measurement, and $A_k$, $B_k$ and $C_k$ are known bounded real-time-varying matrices. Also, we define the following worst-case performance measure for the system (6)-(7):

\[
J(z, w, x, R) = \sup_{(z_k, w_k) \in (z, w)} \left\{ \left( \frac{\| z \|^2}{\| w \|^2 + x_k^T R x_k} \right)^{1/2} \right\},
\]

where $R > R^T > 0$ is a given weighting matrix for the initial state and $0 \neq (z_0, w) \in \mathbb{R}^p \oplus l_2([0,\infty))$. Then, we have the following result.

Lemma 1 [16]: Consider the system (6)-(7) and let $\gamma > 0$ be a given scalar. Then, the following statements are equivalent:

(a) The system (6) is exponentially stable and $J < \gamma$.
(b) There exists a bounded time-varying matrix $Q_k = Q_k^T \geq 0$, $\forall k \geq 0$, satisfying $I - \gamma^2 C_k^T C_k > 0$, $\forall k \geq 0$, and such that

\[
A_k^T Q_k A_k^T - Q_k + \gamma^2 A_k^T C_k (I - \gamma^2 C_k^T C_k)^{-1} C_k A_k = 0,
\]

and the system

\[
x_{k+1} = \{A_k + \gamma^2 A_k C_k (I - \gamma^2 C_k^T C_k)^{-1} C_k\} x_k
\]

is exponentially stable.
(c) There exists a bounded time-varying matrix $P_k = P_k^T > 0$, $\forall k \geq 0$, satisfying $I - \gamma^2 P_k^T P_k > 0$, $\forall k \geq 0$, and such that

\[
A_k^T P_k A_k - P_k + \gamma^2 A_k^T B_k (I - \gamma^2 B_k^T P_k B_k)^{-1} B_k A_k = 0,
\]

and $x_{k+1} = \{A_k + \gamma^2 A_k B_k (I - \gamma^2 B_k^T P_k B_k)^{-1} B_k\} x_k$.
can be viewed as the
and an initial state weighting matrix
and satisfying Assumption 1. Given a scalar
Theorem 1 reduces to the following corollary.
parts of the proof can be easily established similarly to the
H∞ FIR filtering problem is to be dealt with. It is noted that the
problem does not need the assumption of stabilizability or
detectability of the system since it is formulated on the finite
moving horizon.

III. Nonlinear $H_\infty$ FIR filters

In the sequel we shall provide a solution to both the problems of nominal and robust $H_\infty$ filtering with FIR structure using a Riccati equation approach. We first present a performance analysis result for the system (1)-(3).

**Theorem 1:** Consider the system (1)-(3) satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system (1) is globally uniformly asymptotically stable and decrescent. It can be easily shown that $I - \gamma^2 B^T Q_s B > 0$, for all $\gamma > 0$, and such that

\[
\gamma^2 B^T Q_s B > 0, \quad \forall k \geq 0,
\]

where $Q_s = Q_s^T > 0$, $\forall k \geq 0$, satisfying $I - \gamma^2 B^T Q_s B > 0$, $\forall k \geq 0$, and such that

\[
A^T Q_s A - Q_s + \gamma^2 A^T Q_s B (I - \gamma^2 B^T Q_s B)^{-1} B^T Q_s A + L^T L + \varepsilon^2 E^T E + V_1^T V_1 < 0, \quad Q_0 < \gamma^2 R,
\]

where $A = \begin{bmatrix} B & \gamma \gamma G \end{bmatrix}$. 

**Proof:** Define a Lyapunov function candidate $V(x_k) = x_k^T Q_s x_k$. Since $D I < Q_s < D_2^T I$, $V(x_k)$ is positive definite and decrescent. It can be easily shown that $\Delta V(x_k) = V(x_{k+1}) - V(x_k) \leq -\delta x_k^T X_k$ along the trajectory of (1). Hence, $V(x_k)$ is a Lyapunov function and the system (1) is globally uniformly asymptotically stable. Remaining parts of the proof can be easily established similarly to the proof of Theorem 4.2 in [15].

In the case when there is no parameter uncertainty in (1), Theorem 1 reduces to the following corollary.

**Corollary 1:** Consider the system (1)-(3) with $\Delta A_k = 0$ and satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system (1) is globally uniformly asymptotically stable and decrescent. It can be easily shown that $I - \gamma^2 B^T Q_s B > 0$, $\forall k \geq 0$, and such that

\[
A^T Q_s A - Q_s + \gamma^2 A^T Q_s B (I - \gamma^2 B^T Q_s B)^{-1} B^T Q_s A + L^T L + \varepsilon^2 E^T E + V_1^T V_1 < 0, \quad Q_0 < \gamma^2 R,
\]

where $Q_s = Q_s^T > 0$, $\forall k \geq 0$, satisfying $I - \gamma^2 B^T Q_s B > 0$, $\forall k \geq 0$, and such that

\[
A^T Q_s A - Q_s + \gamma^2 A^T Q_s B (I - \gamma^2 B^T Q_s B)^{-1} B^T Q_s A + L^T L + \varepsilon^2 E^T E + V_1^T V_1 < 0, \quad Q_0 < \gamma^2 R,
\]

where $\tilde{B} = [B \gamma G]$. Note that when the initial state of the system (1) is known to be zero, the time-varying matrix $Q_s$ in Theorem 1 and Corollary 1 may be replaced by a constant matrix $Q = Q^T > 0$. Furthermore, the condition $Q < \gamma R$ will no longer be required as an initial state which is certain to be zero corresponds to choosing a `very large' value of $R$.

We now present a solution to the nominal nonlinear $H_\infty$ FIR filtering problem for the system (1)-(3).

**Theorem 2:** Consider the system (1)-(3) with $\Delta A_k = 0$ and $\Delta C_k = 0$, and satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$, the nominal nonlinear $H_\infty$ FIR filtering problem is solvable if there exists a bounded time-varying matrix $S_k = S_k^T > 0$, $\forall k \geq 0$, satisfying $I - \gamma^2 \tilde{L}_k \tilde{E}_k > 0$, $\forall k \geq 0$,

\[
S_{k+1} = AS_k A^T - (AS_k C^T + \tilde{B} \tilde{D}^T) (\tilde{C} S_k C^T + \tilde{R})^{-1} (\tilde{C} S_k A^T + \tilde{D} \tilde{B}^T) + \tilde{B} \tilde{D}^T, \quad S_0 = R^{-1},
\]

and the system

\[
\rho_{k+1} = \rho_k \begin{bmatrix} A - (AS_k C^T + \tilde{B} \tilde{D}^T) (\tilde{C} S_k C^T + \tilde{R})^{-1} \tilde{C} \end{bmatrix} \rho_k
\]

is exponentially stable, where

\[
\tilde{L}_k L = L_k^T L + V_k^T V_k, \quad V = \begin{bmatrix} V_1^T & V_2^T \end{bmatrix}, V_k = \begin{bmatrix} G_k \rho_k \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & \gamma \gamma H_k \end{bmatrix}, \quad \tilde{C}_k = C \begin{bmatrix} C \gamma \gamma I \end{bmatrix}, \quad \tilde{D}_k = D \begin{bmatrix} 0 & \gamma \gamma H_k \end{bmatrix}, \quad \gamma = R^T D \begin{bmatrix} 0 & -I \end{bmatrix}
\]

Moreover, if the above conditions hold, a suitable nonlinear filter is given by

\[
x_{\gamma,k+1} = A x_{\gamma,k} + G_k (x_{\gamma,k}) + K_k [y_k - C x_{\gamma,k} - H_k (x_{\gamma,k})]
\]

where

\[
K_k = (A \tilde{S}_k C^T + \tilde{B} \tilde{D}^T) (\tilde{C} \tilde{S}_k C^T + \tilde{D} \tilde{D}^T)^{-1}
\]

and

\[
\tilde{S}_k = S_k + \gamma^2 S_k L^T (I - \gamma^2 \tilde{L}_k \tilde{E}_k)^{-1} \tilde{L}_k
\]

**Proof:** Firstly, note that the condition $I - \gamma^2 \tilde{L}_k \tilde{E}_k > 0$, $\forall k \geq 0$, together with Assumption 1 guarantees the non-singularity of the matrix $\tilde{R} + \tilde{C} S_k C^T$, $\forall k \geq 0$. Letting $x_k = x_k - x_k$, and $e_k = z_k - z_k$, it follows from (1)-(3) (setting $\Delta A_k = 0$ and $\Delta C_k = 0$) and (11)-(12) that

\[
x_{\gamma,k+1} = (A - K_k C) x_k + (C - K_k H_k) z_k (x_k - x_k) + (B - K_k D) z_k
\]

where

\[
e_k = L_k z_k.
\]
where
\[ \xi(x_k, x_{k+1}) = \begin{bmatrix} g(x_k) - g(x_{k+1}) \\ h(x_k) - h(x_{k+1}) \end{bmatrix} \]
and \( G = \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix} \).

Note that by Assumption 1, \( \|\xi(x_k, x_{k+1})\| \leq V_N \|\xi\| \).

It can be shown from (9) that \( Q_k = R_k \gamma^2 S_k \) is such that \( I - L_k Q_k L_k^T > 0 \), \( \forall k \geq 0 \), and satisfies
\[ (A - K_k C)Q_k (A - K_k C)^T = Q_{k+1} + (A - K_k C)Q_k L_k^T \]
\[ (I - L_k Q_k L_k^T)^{-1} L_k Q_k (A - K_k C)^T + \gamma^2 \tilde{B}_k \tilde{B}_k^T = 0, \]
\[ Q_0 = \gamma^2 R^{-1}, \] (17)
where \( \tilde{B}_k = [(B - K_k D)^\gamma(G - K_k H_1)] \).

Also, it is easy to verify that the state matrix \( A_{th} \) of the system (10) can be rewritten as
\[ A_{th} = (A - K_k C)(I + Q_k L_k (I - L_k Q_k L_k^T)^{-1} L_k^T) \].

Since the system of (10) is exponentially stable, in view of Lemma 1 and Corollary (1) and (5) implies that the estimation error dynamics of (15)-(16) is globally uniformly asymptotically stable and \( \|\xi\| \leq V N \|\xi\| \) for any non-zero \( (x_k, \xi) \in \mathbb{R}^l \otimes \mathbb{L}_{[0,\infty)} \).

When the initial state of the system (1) is known to be zero, and a stationary filter is concerned, Theorem 2 can be simplified as follows.

Theorem 3: Consider the system (1)-(3) with \( x_0 = 0 \), \( \Delta A_k = 0 \), \( \Delta C_k = 0 \), and satisfying Assumption 1. Given a scalar \( \gamma > 0 \) and an initial state weighting matrix \( R = R^T > 0 \), the nominal nonlinear \( H_\infty \) FIR filtering problem is solvable if there exists a stabilizing solution \( S = S^T \geq 0 \), to the algebraic Riccati equation:
\[ S = AS^T - (ASC^T + BDB^T) + B^2 \gamma^2 R \] (18)
such that \( I - \gamma^2 L_k L_k^T > 0 \). Moreover, if the above conditions hold, a suitable nonlinear filter is given by (11)-(12), where the filter gain of (13) is constant.

It should be pointed out that in Theorems 2 and 3 no stability requirement is imposed on the system (1). We also observe that, when there are no nonlinear terms in the system (1)-(2), i.e. \( g(\cdot) \equiv 0 \) and \( h(\cdot) \equiv 0 \), the result of Theorem 3 will reduce to the \( H_\infty \) FIR linear filter.

IV. Robust nonlinear \( H_\infty \) FIR filters

Next, we solve the robust nonlinear \( H_\infty \) FIR filtering problem. To this end, we shall make a further assumption on the system (1).

Assumption 2: The nominal state matrix \( A \) is nonsingular.

Theorem 4: Consider the uncertain system (1)-(3) satisfying (4)-(5) and Assumptions 1-2. Let \( \gamma > 0 \) be an arbitrary small scalar. Given a scalar \( \gamma > 0 \) and an initial state weighting matrix \( R = R^T > 0 \), the robust \( H_\infty \) FIR filtering problem is solvable if for some scalar \( \epsilon > 0 \), the following conditions hold:
(a) There exists a stabilizing solution \( P = P^T > 0 \) to the algebraic Riccati equation:
\[ A^T PA + P + y^2 A^T PB (I - \gamma^2 B^T P B) + \gamma^2 R \]
\[ = E_k^T E_k + \epsilon^2 V_N \]
such that \( I - \gamma^2 B^T P B > 0 \), and \( P < \gamma^2 R \), where \( B_k \) is as in (17) and
\[ E_k^T E_k = \epsilon^2 E^T E + V_N \]
(20)
(b) There exists a bounded time-varying matrix \( S_k = S_k^T \geq 0 \), \( \forall k \geq 0 \), satisfying \( I - \gamma^2 L_k L_k^T > 0 \), \( \forall k \geq 0 \), and such that
\[ S_{k+1} = \hat{A}_k S_k \hat{A}_k^T - (\hat{A}_k \hat{C}_k^T + \hat{B}_k \hat{D}_k^T) (\hat{C}_k S_k \hat{C}_k^T + \hat{R})^{-1} \]
\[ (\hat{C}_k S_k \hat{A}_k^T + \hat{D}_k \hat{D}_k^T) + \hat{R}, \]
\[ S_0 = (R - \gamma^2 P)^{-1} \] (21)
and the system
\[ \phi_{k+1} = A_{th} \phi_k \]
\[ = [(I - (\hat{A}_k S_k \hat{C}_k^T + \hat{B}_k \hat{D}_k^T) (\hat{C}_k S_k \hat{C}_k^T + \hat{R})^{-1} \hat{C}_k \phi_k] \]
(22)
is exponentially stable, where
\[ \hat{L}_k^T = L_k^T + V_k^T V_k, \]
\[ \hat{C}_k = \hat{C}_k \]
\[ \hat{D}_k = [\hat{D}D^T 0], \]
\[ \hat{R} = [R 0 -I] \]
\[ \hat{A} = A + \epsilon^2 \hat{R} \]
\[ \hat{C} = C + \epsilon^2 \hat{R}\hat{C} \]
\[ \hat{B} = [\hat{B}M \hat{C}G \hat{B}] \]
\[ \hat{D} = [\hat{D} M 0 \gamma H], \]
\[ M = [I + \gamma^2 \hat{R}^2 (P - I - \gamma^2 B_k B_k^T) \hat{B}^T] \]
(30)
Moreover, if conditions (a) and (b) are satisfied, a suitable nonlinear filter is given by
\[ x_{\phi(k+1)} = \hat{A}_k x_{\phi} + G \hat{g}(x_{\phi}) + K_k [y_k - \hat{C}_k x_{\phi} - \hat{H} \hat{x}_{\phi}], \]
\[ z_{\phi} = L x_{\phi}, \]
(32)
where
\[ K_k = (\hat{A}_k \hat{C}_k^T + \hat{B}_k \hat{D}_k^T) (\hat{C}_k \hat{S}_k \hat{C}_k^T + \hat{D}_k \hat{D}_k^T) \]
\[ \hat{S}_k = S_k^T - \gamma^2 L_k^T (I - \gamma^2 L_k^T)^{-1} L_k \]
(34)
Proof: First, note that since \( P > 0 \) and \( I - \gamma^2 B^T P B > 0 \), it follows that the matrix \( P - I - \gamma^2 B^T P B \) is positive definite. Hence, the coefficient matrices of (21) are well defined. Moreover, the condition \( I - \gamma^2 M S_k \hat{A}_k^T > 0 \), \( \forall k \geq 0 \), together with Assumption 1(a) guarantee the non-singularity of \( \hat{C}_k \hat{S}_k \hat{C}_k^T + \hat{R} \) for all \( \forall k \geq 0 \).
Note that as $P$ is the stabilizing solution of (19), $\bar{A}$ is Schur stable. Moreover, since the system (22) is exponentially stable, it follows that the system (41) is exponentially stable as well. Hence, $\tilde{X}_k$ is the stabilizing solution of (40).

By Lemma 1, this implies that there exist a scalar $\delta > 0$ and a bounded time-varying matrix $Y_k = Y_k^T > 0, \forall k \geq 0$, satisfying $I - \tilde{B}_k^T \bar{Y}_k \tilde{B}_k > 0, \forall k \geq 0$, and such that

$A_k', Y_k + A_k' Y_k \tilde{B}_k (I - \tilde{B}_k^T \bar{Y}_k \tilde{B}_k)^{-1} \tilde{B}_k^T \bar{Y}_k A_k$

$+ C_k^T \tilde{C}_k < 0, \quad Y_k < \bar{R}.

Now, taking into account that

$\tilde{C}_k^T \tilde{C}_k = C_k^T C_k + \varepsilon^2 E_k^T E_k + \tilde{V}^T \tilde{V} + \begin{bmatrix} 0 & V \end{bmatrix}$

we obtain that $Y_k$ satisfies the following inequality

$A_k' Y_k + A_k' Y_k \tilde{B}_k (I - \tilde{B}_k^T \bar{Y}_k \tilde{B}_k)^{-1} \tilde{B}_k^T \bar{Y}_k A_k$

$+ C_k^T C_k + \varepsilon^2 E_k^T E_k + \tilde{V}^T \tilde{V} < 0, \quad Y_k < \bar{R}.

Also, note that $\eta_0^T \bar{R}_{10} = \gamma^2 x_0^T R x_0$. Finally, in view of the definition of $\tilde{B}_k$, it follows from Theorem 1 that the estimation error system (37)-(38) is globally uniformly asymptotically stable and

$\|e\| < \gamma \|w\|_1 + \gamma^2 R x_0 \gamma^{1/2}$

for all non-zero $(x_0, w) \in \mathbb{R}^n \otimes \mathbb{L}_1 [0, \infty)$ and for all admissible uncertainties.

The arbitrary small scalar $\gamma > 0$ is introduced in Theorem 4 to guarantee that the stabilizing solution of (19) is positive definite. In the case when $E_k^T E > 0$, or the pair $(A, E_k)$ has no unobservable modes in the closed unit disk, $\gamma$ can be set to zero.

We observe that the existence of a matrix $P$ satisfying condition (a) of Theorem 4 will guarantee the global uniform asymptotic stability of the uncertain system (1) for all uncertainties satisfying (4)-(5). Note that due to the existence of parameter uncertainty in (1), the requirement of global asymptotic stability of (1) is needed in order to ensure the boundedness of the estimation error dynamics for all admissible uncertainties.

It should be noted that the result of Theorem 4 does not recover that of Theorem 2 when the uncertainties $\Delta A_k$ and $\Delta C_k$ disappear. The reason for this is because when parameter uncertainty exists an asymptotic stability requirement has to be imposed on the system of (1), which gives rise to (19) of Theorem 4.

V. Example

To demonstrate the use of the above theory we consider the robust nonlinear $H_\infty$ FIR filter for a simple second-order problem. We show the advantage of the proposed technique by comparing its results with the corresponding results of the $H_\infty$ nonlinear estimator of Shaked and Berman [13] and the extended Kalman filter (EKF), which has been widely used in the past in estimation of nonlinear systems.

Consider the time-invariant process with a saturating nonlinear estimator of Shaked and Berman [13] and the extended Kalman filter (EKF), which has been widely used in the past in estimation of nonlinear systems.

$X_{k+1} = (A + \Delta A_k)x_k + G(x_k)$

$+ K_k [Y_k - (C + \Delta C_k)x_k - H h(x_k)]$ \hspace{1cm} (35)

$z_k = M x_k,$ \hspace{1cm} (36)

where $\Delta A_k$ and $\Delta C_k$ are given in (25)-(26), respectively. We note that $\Delta A_k$ and $\Delta C_k$ reflect the effect of the parameter uncertainties $\Delta A_k$ and $\Delta C_k$ on the filter structure. When $\Delta A_k$ and $\Delta C_k$ in the system (1)-(3) disappear, $\Delta A_k$ and $\Delta C_k$ in (35) will naturally be set to zero.

Defining $\bar{x} = x - x_0$, from (1)-(3) and (35)-(36) we obtain that

$\eta_{k+1} = (A + \Delta A_k) \bar{x}_k + G \bar{x}_k + B \bar{w}_k$

$e_k = C \eta_k, \quad \bar{e} = [\bar{x}^T, \bar{z}^T]^T,$ \hspace{1cm} (37)

where $\bar{e} = z - z_0$.

$A_k = \begin{bmatrix} A & 0 \\ - (\Delta A_k - K \Delta C_k) & A + \Delta A_k - K \Delta C_k \end{bmatrix},$

$B_k = \begin{bmatrix} B \\ B - K_k D_k \end{bmatrix}, \quad H_k = \begin{bmatrix} I_l \\ I_l - K_k L_k \end{bmatrix},$

$C_k = \begin{bmatrix} G & 0 \\ 0 & G - K_k H_k \end{bmatrix}.$

$g_1(x_k), g_2(x_k), h_1(x_k), h_2(x_k)$

Note that by Assumption 1 we have that

$\|g_1(x_k, x_{k+1})\| < \|\tilde{V}\eta_k\|$

where $\tilde{V} = \text{diag}(V, V)$ and $V$ is as in (23).

Next, it can be shown by using standard, but tedious, matrix manipulations that

$X_k = \begin{bmatrix} P^{-1} & 0 \\ 0 & \gamma^{-2} S_k \end{bmatrix}$ \hspace{1cm} (39)

where $P$ and $S_k$ are the required solutions of (19) and (21) satisfies

$A_k S_k A_k^T - X_{k+1} + A_k X_k \tilde{C}_k (I - \tilde{C}_k^T X_k \tilde{C}_k)^{-1} \tilde{C}_k X_k A_k^T + \tilde{B}_k \tilde{B}_k = 0, \quad X_0 = \bar{R}^{-1},$ \hspace{1cm} (40)

where $\bar{R} = \text{diag}(P, \gamma^2 R - P)$.

We now show that $X_k$ given in (39) is such that the time-varying system

$\rho_{k+1} = \hat{A} \rho_k = [A_k + A_k X_k \tilde{C}_k (I - \tilde{C}_k^T X_k \tilde{C}_k)^{-1} \tilde{C}_k] \rho_k$ \hspace{1cm} (41)

is exponentially stable. It can be shown that

$\hat{A} = \begin{bmatrix} A & 0 \\ \ast & A_{2k} \end{bmatrix},$

where $A_{2k}$ is as in (22), $\ast$ denotes entries which are bounded but irrelevant, and

$\bar{A} = A + \gamma^{-2} B_k (I - \gamma^{-2} B_k^T P B_k)^{-1} B_k^T PA.$
nonlinearity in the system dynamics

\[
\begin{bmatrix}
\dot{x}_{\text{est}} \\
\dot{x}_{\text{est}}
\end{bmatrix} =
\begin{bmatrix}
\mu x_i \\
\tan^{-1}(0x_i + \lambda x_i)
\end{bmatrix} +
\begin{bmatrix}
0.4 \\
0.4
\end{bmatrix} F(x_i) E(x_i) +
\begin{bmatrix}
0.5 \\
1.5
\end{bmatrix} w_i .
\] (43)

where \(\mu = 0.91, \eta = -0.07, \lambda = 0.1, E(x_i) = [0 \ 1] x_i , \) and \(|F(x_i)| \leq 1, \forall i \in [0, N] \) and \(\forall x_i \in \mathbb{R}^2 .\)

We assume here that the measurement is described by

\[ y_i = \cos(2x_{i2}) + 3x_{i2} + 2F(x_i) E(x_i) + 0.01 w_i . \] (44)

We consider the time interval \(N = 500, \) and we are looking for an estimate of \(Lx_i, \) where \(L = [1 \ 0] .\) We also assumed that \([w_i] \) are uncorrelated standard gaussian white noise processes for the comparison of simulation results between robust nonlinear \(H_\infty \) FIR filter and EKF.

**Fig. 1. Estimation error of the robust \(H_\infty \) FIR filter.**

**Fig. 2. Estimation error of the robust \(H_\infty \) filter.**

**Fig. 3. Estimation error of the extended Kalman filter.**

We have simulated the above three estimators for the worst values of the uncertainty \(F, \) namely for each estimator we describe the estimators. Fig. 1, 2 and 3 show the estimation error that have been obtained for the three estimators, where \(F = 1 \) for the robust nonlinear \(H_\infty \) FIR filter, the robust nonlinear \(H_\infty \) filter of Shaked and Berman [13] and the EKF and \(\gamma \) for the robust nonlinear \(H_\infty \) FIR filter and robust nonlinear \(H_\infty \) filter are 6.3 and 1.1, respectively. Note that estimation error covariances of the proposed nonlinear \(H_\infty \) FIR filter, the robust nonlinear \(H_\infty \) filter and the extended Kalman filter are \(-2.7307e-004, 1.5995e-002 \) and \(1.4377e-002, \) respectively. These results exemplify that the estimation performance of the robust \(H_\infty \) FIR filter is better than those obtained by the robust nonlinear \(H_\infty \) filter and by the Extended Kalman filter.

**VI. Conclusions**

In this paper the robust nonlinear \(H_\infty \) FIR filter has been proposed for nonlinear discrete time-varying systems with parameter uncertainty. Firstly, the discrete-time \(H_\infty \) FIR filter is obtained for the nonlinear system without the parametric uncertainty. Secondly, the robust nonlinear \(H_\infty \) FIR filter for the uncertain discrete-time nonlinear systems is derived in the modified system model. This paper is an extension of previous works by Kwon et al. [10][11] to nonlinear system, which treat linear time-varying systems with parameter uncertainty.

**References**

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