Position Control of Chained Multiple Mass-Spring-Damper Systems – Adaptive Output Feedback Control Approaches

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Abstract: This paper addresses the issue of position control of a chain of multiple mass-spring-damper (CMMSD) units which can be found in many physical systems. The dynamic model of a CMMSD system with any degrees of freedom is expressed in a closed-form for the convenience of the controller design. Backstepping and model reference adaptive control (MRAC) approaches are then used to develop two adaptive output feedback controllers to control the position of a CMMSD system. The proposed controllers rely on the measurements of the input (force) and the output (position of the mass unit at the end of the chain) of the system without the knowledge of its parameters and internal states. Simulations are used to verify the effectiveness of the controllers.

Keywords: Adaptive control, output feedback, position control, robotics.

1. INTRODUCTION

Chained multiple mass-spring-damper (CMMSD) units are found in many physical systems such as hyper-redundant mechanical systems [1], flexible link Robots [2-5] and multi-mass systems for vibration absorbers [6], to name a few. The position control of a CMMSD system is challenging due to the difficulties in measuring its system parameters and internal states which are “hidden” in a chain of mass-spring-damper units. One typical example is the control of a robot constrained by a flexible constraint where the measurement of the constraint states are almost impossible [4]. It was proved that the model free linear controller such as PID control is not effective to control a CMMSD system [2,7,8]. Though a CMMSD system model can be transformed into a group of decoupled integrators and the controller design can be simplified through the feedback linearization, it needs undesirable high order derivatives of the system states and introduces unstable internal dynamics into the controlled system [2,7]. Though singular perturbation is a powerful tool to make the controller design simpler and effective for CMMSD systems [7], the requirement of very large stiffnesses inside the CMMSD system limited its applications.

To control an uncertain CMMSD system when only the input (the force acting at the end of the chain) and the output (position of the mass unit at another end of the chain) are measurable, adaptive output feedback control should be investigated. The dynamic models of the CMMSD systems do not provide the matching condition, thus a lengthy state transformation is needed before the traditional non-recursive adaptive output feedback control approaches, such as model reference adaptive control (MRAC) can be applied [15]. On the other hand, CMMSD systems are recursive in their physical structures and their dynamic models can be expressed in the parametric strict feedback forms suitable for backstepping design [9]. Though CMMSD systems of low degrees of freedom are fairly simple in structure, control system design for the CMMSD system of high degrees of freedom is still worth investigation owing to the difficulties explained. As one of the contributions of the paper, a closed-form description is given for a CMMSD system of arbitrary degrees of freedom. To the best of our knowledge, this is the first time in the literature a closed form expression of the system model is reported for a general CMMSD system. In fact, the closed-form description lays the foundation for the design of two neat and concise adaptive output feedback controls. The beauty of the controllers presented lies in their clarity and elegant applications of existing theories in solving practical problems,

Fig. 1. A general chained multiple mass spring system (CMMSD).
rather than fundamental contributions in theory. Both backstepping and MRAC methods are used to develop two adaptive output feedback controllers respectively to control the position of CMMSD systems without special requirements on the system parameters (unlike singular perturbation approach which requires sufficiently large stiffnesses of the springs in the CMMSD system), without the need for undesirable high derivative of system states and avoid unstable internal dynamics. The asymptotical stability of the controlled system is guaranteed under the controllers proposed. The main contributions of the paper are as follows:

• the establishment of a closed form description for a general CMMSD system with arbitrary any number of mass-spring-damper units with any stiffness of the spring, and

• the development of two elegant adaptive output feedback position controllers – backstepping based and MRAC based, that are independent of the system parameters and undesirable high derivative of system states.

The rest of paper is organized as follows. In Section 2, the dynamic model of the system is given. In Section 3, two adaptive output feedback controllers are developed with backstepping and MRAC approaches respectively. In section 4, the simulation study is done to verify and compare the effectiveness of the approaches proposed. The conclusion is given in Section 5.

2. DYNAMIC MODELING AND PROBLEM FORMULATION

A chained multiple mass spring system with \( n \) mass units is schematically shown in Fig. 1, where \( m_i \) is the mass, \( b_i \) is the viscous coefficient and \( q_i \) is the displacement measured from the equilibrium position along the \( X \) axis of the \( i \) th unit \( (i=1,2...n) \). There are \( n-1 \) springs connecting the mass units with \( k_i \) being the linear spring constant \( (i=1,2...n-1) \). It is understood that only \( q_1 \), the output of the system, and \( u \), the input force, are measurable.

According to Newton’s second law, we have the following system dynamics:

\[
\begin{align*}
\dot{m}_i \ddot{q}_i &= -b_i \dot{q}_i + k_i (q_2 - q_i) = -b_i \dot{q}_i - k_i q_1 + k_i q_2, \\
\dot{m}_2 \ddot{q}_2 &= -b_2 \dot{q}_2 -(k_1 + k_2) q_2 + k_1 q_1 + k_2 q_3, \\
\dot{m}_3 \ddot{q}_3 &= -b_3 \dot{q}_3 -(k_{i-1} + k_i) q_i + k_{i-1} q_{i-1} + k_i q_{i+1}, \\
\dot{m}_n \ddot{q}_n &= -b_n \dot{q}_n - k_{n-1} q_{n-1} + k_n q_{n-1} + u, \\
\end{align*}
\]

\( i = 3,4,...,n-1 \).

Defining \( x_1 = q_1, \ x_2 = \dot{q}_1, \ x_3 = q_2, \ x_4 = \dot{q}_2, \ ..., \ x_{2i-1} = q_i, \ x_{2i} = \dot{q}_i, \ ..., \ x_{2n-1} = q_n, \ x_{2n} = \dot{q}_n, \)

\( x=[x_1 \ x_2 \ ... \ x_{2n}]^T \in \mathbb{R}^{2n} \), the original dynamic system (1) is transformed to the following state space model:

\[
\begin{align*}
\dot{x} &= A_x x + b_x u, \\
x_1 &= c_1^T x,
\end{align*}
\]

where

\[
\begin{align*}
b_x &= m_n^{-1} c_{2n}, \\
A_x &= [A_1^T \ A_2^T \ A_3^T \ ... \ A_{2n-1}^T \ A_{2n}^T]^T, \\
A_1 &= c_2^T, \\
A_2 &= -m_1^{-1}(k_1 c_1^2 + b_2 c_1^2 - k_1 c_2^2), \\
A_{2i-1} &= c_{2i}^T, \\
A_{2i} &= m_i^{-1} (k_{i-1} c_{2i-1}^2 - (k_i + k_{i+1}) c_{2i}^2 - b_i c_{2i}^2) \\
&\quad + m_{i+1}^{-1} k_i c_{2i+1}^2 (i=2,3,...,n-1), \\
A_{2n-1} &= c_{2n}^T, \\
A_{2n} &= m_n^{-1}(k_{n-1} c_{2n-1}^2 - k_n c_{2n}^2 - b_n c_{2n}^2), \\
\end{align*}
\]

and \( c_j \) is the \( j \)th column vector of identity matrix \( I_{2n} \) \( (j=1,2,...,2n) \).

Applying Laplace transformation on equations (2) and (3), the system dynamic model in the \( s \) domain is obtained:

\[
X_1(s) = H_{2n}(s) U(s), \quad (4)
\]

where \( X_1(s) \) and \( U(s) \) are the Laplace transformation of \( x_1 \) and \( u \) respectively, and \( H_{2n} \) is the transfer function defined as

\[
H_{2n}(s) = c_1^T (sI_{2n} - A_1)^{-1} b_x = \frac{d_{2n}}{s^{2n} + \sum_{j=0}^{2n-1} a_{2n-j} s^j}. \quad (5)
\]

The coefficients in the transfer function are expressed as the explicit functions of the system parameters in Appendix 1. For clarity, their detailed and involved derivations are omitted.

Remark 1: For a CMMSD of arbitrary any degrees of freedom, we have obtained a closed-form description (5) that is explicitly expressed as the functions of the system parameters. This closed-form expression is not only essential for the control design discussed in this paper, but also very convenient and useful for simulation and system analysis.

Re-writing the state space equations in observer canonical form for system (4), we have

\[
\begin{align*}
y &= A y + B (y_1, u)^T \theta, \\
y_1 &= x_1 = c_1^T y,
\end{align*}
\]

where
\[ \dot{y}_1 = y_2 - a_{2n,2n-1} y_1 \]
\[ \vdots \]
\[ \dot{y}_i = y_{i+1} - a_{2n,2n-i} y_1 \]
\[ \vdots \]
\[ \dot{y}_{2n} = d_{2n} u \]
\[ A = \begin{bmatrix} 0 & I_{2n-1} \\ 0 & \ldots & 0 \end{bmatrix} \]
\[ B^T = [c_{2n}u - I_{2n}y_1] \]
\[ \theta = [d_{2n} a_{2n,2n-1} a_{2n,2n-2} \ldots a_{2n,1} a_{2n,0}]^T. \]

Note that \( B \in R^{(2n+1) \times 2n} \) is a matrix formed by the input \( u \) and the output \( y_1 \), and \( \theta \in R^{2n+1} \) is a vector of the coefficients \( d_{2n} \) and \( a_{2n,j} \) (\( j = 0, 1, 2, \ldots, 2n \)).

**Remark 2:** It is essential to re-write the system model in observer canonical form (6) so that the states can be reconstructed for output feedback control design that follows. As clearly shown, this is only possible after the closed-form description (5) is obtained.

## 3. CONTROLLER DESIGN

In this section, adaptive output feedback is investigated based on the observer canonical form (6) of the CMMSD system. The control objective is to regulate the output \( y_1 \) of the CMMSD system. The control objective is to investigate based on the observer canonical form (6) is obtained.

### 3.1. Adaptive output feedback control using backstepping

First, let us consider the following \( \mathcal{K} \)-filters [9]:
\[ \dot{\xi} = A_0 \xi + \lambda y_1, \]  
\[ \Omega^T = A_0 \Omega^T + B(y_1,u)^T, \]  
where \( \xi \in R^{2n} \) and \( \Omega^T \in R^{2n \times (2n+1)} \) are the outputs of the filters, and \( \lambda = [\lambda_1 \lambda_2 \ldots \lambda_{2n}]^T \in R^{2n} \) are parameters that are chosen such that
\[ A_0 = A - \lambda e_1^T \in R^{2n \times 2n} \]
\[ PA_0 + A_0^T P = -Q < 0 \]  
for any given symmetric positive definite matrices \( P \in R^{2n \times 2n} \) and \( Q \in R^{2n \times 2n} \). For simplicity and clarity, we will take \( Q = I_{2n} \) in the paper.

To reduce the order of the filters, \( \Omega^T \) is chosen as
\[ \Omega^T = [v \ \Omega_2], \]
where  
\[ v = [v_1 \ v_2 \ldots v_{2n}]^T \in R^{2n}, \]
\[ \Omega_2 = [\eta_1 \ \eta_2 \ldots \eta_{2n}] \in R^{2n \times 2n}, \]
and \( \eta_j \in R^{2n} \) (\( j = 1, 2, \ldots, 2n \)).

With \( v \) and \( \Omega_2 \) so defined, we have
\[ \dot{v} = A_0 v + c_{2n} u, \]
\[ \Omega_2 = A_0 \Omega_2 - I_{2n} y_1. \]  

Due to the special structure of \( A_0 \) and from equations (7) and (10), we have
\[ \eta_{2n} = A_0 \eta_{2n} - c_{2n} y_1, \]
\[ \eta_j = A_0^{2n-j} \eta_{2n}, \]
\[ \xi = A_0^{2n} \eta_{2n}. \]

With \( \mathcal{K} \)-filters determined above, the estimate of \( y \) is given by
\[ \dot{\hat{y}} = \dot{\xi} + \Omega^T \hat{v}. \]  

It can be shown that the state estimation error follows
\[ \dot{\epsilon} = A_0 \epsilon. \]

Defining a Lyapunov function
\[ V_\epsilon = \epsilon^T P \epsilon \]
and differentiating it with respect to time \( t \), since \( A_0 \) is symmetric we have
\[ \dot{V}_\epsilon = 2 \epsilon^T P \dot{\epsilon}. \]

From equations (9) and (13), it can be concluded that
\[ \dot{V}_\epsilon = -2 \epsilon^T P \epsilon \]
and \( \epsilon \) approach zero while \( t \to \infty \). From equations (11) and (12), and considering equations (6) and (8), we have
\[ \dot{v}_1 = c_{2n} A_0^{2n} \eta_{2n} + w^T \theta + \epsilon_2, \]
\[ \dot{v}_2 = v_3 - \lambda_2 \eta_1, \]
\[ \dot{v}_i = v_{i+1} - \lambda_i \eta_{i-1} \quad (i = 3, 4, \ldots, 2n - 1), \]
\[ \dot{v}_{2n} = -\lambda_{2n} \eta_{2n} + u, \]
where
\[ w = [v_2 \ \eta_{2n} A_{0n}^T - y_1 c_{1}^T]^T, \]
\[ \bar{w} = 0 \]
\[ A_{\eta} = [(A_{0n}^{-1})^T c_2 \ldots A_0^T c_2 c_2]^T. \]
Equations (15) to (17) represent a transformed dynamic system with the measurable \( v \) and \( y_1 \) being its states. To facilitate the controller design with backstepping method, we also need the following coordinate transformation:

\[
z_1 = y_1,
\]

\[
z_i = v_i - a_{i-1}, \quad i \geq 2,
\]

\[
z = [z_1, z_2, \ldots, z_{2n}]^T,
\]

where \( z_i \) is the new state and \( q_i \) is the so called stabilization function or virtual control to be determined in the steps of the controller design.

The backstepping design involves \( 2n \) steps. A Lyapunov function is constructed, and from which, a stabilizing function \( \alpha_i \) is determined in every step.

In addition, a function called tuning function \( \tau_i \) is also generated in each step to estimate the uncertain system parameters. In the last step \( 2n \), the control input \( u \) is derived.

**Step 1.** In this step, we begin with the study of the tracking error \( z_1 \). Its equation can be derived from equations (15), (18) and (19), such that

\[
\dot{z}_1 = d_{2n} \alpha_1 + c_2^T A_0^{2n} \eta_2n + \gamma \dot{\theta} - d_{2n} z_2 + e_2.
\]

Considering the following virtual control:

\[
\alpha_1 = \dot{d} \gamma \alpha_1,
\]

where \( d \) is the estimate of \( 1/d_{2n} \), and substituting it into equation (20) leads to

\[
\dot{z}_1 = \alpha_1 + c_2^T A_0^{2n} \eta_2n + \gamma \dot{\theta} - d_{2n} (\dot{d} \gamma \alpha_1 - z_2) + e_2.
\]

Consider the Lyapunov function candidate:

\[
V_1 = \frac{1}{2} \dot{z}_1^2 + \frac{1}{2} \dot{\theta} \Gamma^{-1} \dot{\theta} + \frac{d_{2n}}{\gamma} \dot{d} \dot{z}_2 + V_e,
\]

where \( \dot{\theta} = \dot{\theta} - \dot{\theta} \), \( \dot{\theta} \) is the estimate of \( \theta \), \( \Gamma = \Gamma^{-1} > 0 \), \( \gamma > 0 \) are the gain matrix and gain respectively, and \( V_e \) is defined in equation (14).

Note that it has been proved that \( \dot{V}_e = -\|e\|^2 \).

Differentiating \( V_1 \) with respect to time \( t \), we have

\[
\dot{V}_1 = z_1 \dot{z}_1 - \dot{\theta} \Gamma^{-1} \dot{\theta} - \frac{d_{2n}}{\gamma} \dot{d} \dot{z}_2 - \|e\|^2.
\]

It can be easily verified that along the solution of (22),

\[
\dot{V}_1 \leq -\zeta_1 z_1^2 + \frac{c_1}{2} \dot{\theta} z_1 z_2 + \theta^T (\tau_1 - \Gamma^{-1} \dot{\theta}) - \frac{1}{2} (z_1 - e_2)^2 + \frac{1}{2} e_2^2 - \|e\|^2
\]

\[
\leq -\zeta_1 z_1^2 + c_1 \dot{\theta} z_1 z_2 + \theta^T (\tau_1 - \Gamma^{-1} \dot{\theta})
\]

if we choose

\[
\bar{a}_1 = -(\zeta_1 + \frac{1}{2}) z_1 - c_2^T A_0^{2n} \eta_{2n} - \bar{w} \dot{\theta},
\]

\[
\tau_1 = w - \bar{d} \Gamma \alpha_1,
\]

where \( \zeta > 0 \) is a control parameter and \( \dot{\theta} \) is the estimate of parameters \( \theta \).

Note that if \( v_2 = \alpha_1 \) is the actual control and it leads to

\[
V_1 \leq -\zeta_1 z_1^2 \leq 0.
\]

**Step 2.** From equations (16), (19) and (21), we have

\[
z_2 = \alpha_2 + z_3 - \gamma (w^T \dot{\theta} + e_2) - \frac{d}{\partial \theta} \dot{\gamma} \alpha_1 \dot{\theta} - \beta_2,
\]

\[
\gamma_2 = \frac{d}{\partial \gamma} \gamma \alpha_1,
\]

\[
\beta_2 = \lambda_2 \dot{v}_1 + \frac{d}{\partial \gamma} \gamma \alpha_1 (A_0 \eta_{2n} - c_{2n} y_1) - \gamma \dot{\gamma}_2 \dot{z}_2.
\]

Equation (25) describes the behavior of the tracking error \( z_2 \). To find out the virtual control \( \alpha_2 \) to stabilize \( z_2 \), consider the following Lyapunov function candidate:

\[
V_2 = V_1 + \frac{1}{2} z_2^2 + V_e.
\]

Differentiating \( V_2 \) with respect to time \( t \) along the solutions of (22) and (25), we have

\[
\dot{V}_2 \leq -\zeta_2 z_2^2 + z_2 z_3 + \theta^T (\tau_2 - \Gamma^{-1} \dot{\theta})
\]

\[
+ z_3 (\alpha_2 + c_1 \dot{\theta} z_1 - \beta_2 - \frac{d}{\partial \theta} \dot{\gamma}_2 \alpha_1) - \gamma \dot{\gamma}_2 z_2 e_2 - \|e\|^2,
\]

where \( \tau_2 = \tau_1 - \gamma_2 w e_2 \).

If we select

\[
\alpha_2 = -(\zeta_2 + \frac{1}{4}) z_2 - c_1 \dot{\theta} z_1 + \beta_2 + \frac{d}{\partial \theta} \dot{\gamma}_2 \Gamma \tau_2, \quad \zeta_2 > 0,
\]

it follow that

\[
\dot{V}_2 \leq -\zeta_2 z_2^2 - \zeta_2 z_2^2 + z_2 z_3 + \theta^T (\tau_2 - \Gamma^{-1} \dot{\theta})
\]

\[
+ z_2 \theta \frac{d}{\partial \theta} \dot{\gamma}_2 \Gamma \tau_2 - \dot{\theta}.
\]

If \( v_3 = \alpha_2 \) is the actual control and

\[
z_3 = 0, \quad \dot{\theta} = \Gamma \tau_2,
\]

we have

\[
\dot{V}_2 \leq -\zeta_2 z_2^2 - \zeta_2 z_2^2 \leq 0.
\]
Step \( i (3 \leq i \leq 2n - 1) \) In this step, we generalize the design procedure for any step \( i \geq 3 \) with the assumption that stabilizing functions \( a_i, a_2, \ldots, a_{i-1} \) and tuning functions \( \tau_1, \tau_2, \ldots, \tau_{i-1} \) are derived in previous steps.

It can be shown that the tracking error \( z_i \) satisfies

\[
\dot{z}_i = a_i + z_{i+1} - \gamma_i (w^T \dot{\theta} + e_2) - \frac{\partial a_{i-1}}{\partial \theta} \dot{\theta} - \beta_i,
\]

where

\[
\gamma_i = \frac{\partial a_{i-1}}{\partial \gamma_i},
\]

\[
\beta_i = \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial \gamma_j} (v_{j+1} - \lambda_j \dot{\gamma}_j) + \gamma_i (w^T \dot{\theta} + c_2 \hat{A}_0 \eta_{2n})
\]

\[
+ \lambda_j \dot{\gamma}_j + \frac{\partial a_{i-1}}{\partial \gamma_{2n}} (A_0 \eta_{2n} - c_2 \eta_{2n}) - \frac{\partial a_{i-1}}{\partial \theta} \gamma_i \tau_{i-1}.
\]

Consider the Lyapunov function candidate:

\[
V_i = V_{i-1} + \frac{1}{2} z_i^2 + V_e.
\]

Following the same procedure as in previous steps, the derivate of \( V_i \) with respective to time \( t \) along the solution of equation (27) is rendered as

\[
\dot{V}_i = -\sum_{j=i}^{i-1} \gamma_j z_j^2 + z_i z_{i+1} + \theta_i (\tau_i - \Gamma i \dot{\theta})
\]

\[
+ \sum_{j=2}^{i-1} z_j \frac{\partial a_{i-1}}{\partial \theta} (\Gamma \dot{\theta} - \dot{\theta})
\]

by selecting

\[
a_i = -(\zeta_i + \frac{\gamma_i^2}{4}) z_i - z_{i+1} + \beta_i + \frac{\partial a_{i-1}}{\partial \theta} \Gamma \dot{\theta}
\]

\[
- \sum_{j=2}^{i-1} \frac{\partial a_{i-1}}{\partial \gamma_j} \Gamma \dot{\gamma}_j w,
\]

where \( \zeta_i > 0 \) is a control parameter and \( \tau_i = \tau_{i-1} - \gamma_i w z_i \).

Obviously, if \( v_{i+1} = a_i \) is the actual control and \( z_{i+1} = 0, \dot{\theta} = \Gamma \dot{\theta} \), we have

\[
\dot{V}_i \leq -\sum_{j=1}^{i-1} \gamma_j \zeta_j z_j^2 \leq 0.
\]

Step \( 2n \). This is the final step of backstepping control design where the control input is determined. Following the same way as done for \( a_i \) in the previous steps, the control input can be set as

\[
u = a_{2n} = -\zeta_{2n} z_{2n} - z_{2n-1} + \beta_{2n} + \frac{\partial a_{2n-1}}{\partial \theta} \Gamma_2 \dot{\theta}
\]

\[
- \sum_{j=2}^{2n-1} z_j \frac{\partial a_{2n-1}}{\partial \theta} \Gamma_2 \dot{\gamma}_j w, \zeta_{2n} > 0.
\]

With control input \( u \) in equation (28), parameter updating laws for \( \dot{a} \) and \( \dot{\theta} \) in equations (24) and (33) respectively, combined with \( a_i \) and \( \dot{\theta} - \Gamma \dot{\theta} \) in each step, the resulting close loop system with state vector \( [z_1 z_2 \cdots z_{2n}]^T \) can be described by

\[
\dot{z}_1 = -(\zeta_1 + \frac{1}{2}) z_1 + c_1 \dot{\theta} z_2 + e_2 + w^T \dot{\theta} - d_{2n} \alpha \dot{\theta},
\]

\[
\dot{z}_2 = -c_1 \dot{\theta} z_2 - (\zeta_2 + \frac{\gamma_2^2}{4}) z_2 + z_3
\]

\[
+ \delta \frac{\partial a_{2n-1}}{\partial \gamma_{2n}} \sum_{j=3}^{2n} \Gamma \dot{\gamma}_j w - \gamma_2 (w^T \dot{\theta} + e_2),
\]

\[
\dot{z}_i = -\sum_{j=2}^{i-1} \frac{\partial a_{i-1}}{\partial \theta} \Gamma_{i-1} \dot{\gamma}_j w - \gamma_i (w^T \dot{\theta} + e_2) \quad 3 \leq i \leq 2n - 1,
\]

\[
\dot{z}_{2n} = -\sum_{j=2}^{2n-1} \frac{\partial a_{2n-1}}{\partial \theta} \Gamma_{2n} \dot{\gamma}_j w
\]

\[- (1 + \frac{\partial a_{2n-2}}{\partial \gamma_{2n}}) \Gamma_{2n} \dot{\gamma}_j w z_{2n-1}
\]

\[- (\zeta_{2n} + \frac{\gamma_{2n}^2}{4}) z_{2n} - \gamma_{2n} (w^T \dot{\theta} + e_2).
\]

To prove the asymptotic stability of the closed-loop system represented from equations (29) to (32), consider the following Lyapunov function candidate:

\[
V_{2n} = V_{2n-1} + \frac{1}{2} z_{2n}^2 + V_e
\]

\[
= \frac{1}{2} z_{2n}^2 + \frac{1}{2} \dot{\theta} \Gamma_{2n} \dot{\theta} + \frac{d_{2n}}{2} \dot{\theta}^2 + 2 n V_e.
\]

Differentiating \( V_{2n} \) with respective to time \( t \) along the solutions of (29) to (32), we have

\[
\dot{V}_{2n} \leq -\sum_{j=1}^{2n} \zeta_j z_j^2 + \theta (\tau_{2n} - \Gamma \dot{\theta})
\]

\[
+ \sum_{j=2}^{2n} \dot{\gamma}_j w \frac{\partial a_{2n-1}}{\partial \theta} (\Gamma \dot{\theta} - \dot{\theta})
\]

where \( \tau_{2n} = \tau_1 - \sum_{j=2}^{2n} \dot{\gamma}_j w_j \) and \( \gamma_1 = -1 \) is a new constant introduced to keep the consistency in expression.

Letting

\[
\dot{\theta} = \Gamma \dot{\theta}
\]

it follows that

\[
V_{2n} \leq -\sum_{j=1}^{2n} \zeta_j z_j^2 \leq 0.
\]

From equation (33), it can be concluded that \( V_{2n} \) is
non-increasing. Hence, \( z, \bar{\theta}, \bar{d} \) and \( \varepsilon \) are all bounded. From their boundedness, it can be proved that other signals in the control loop are also bounded. Based on LaSalle-Yoshizawa theorem \([9]\), \( z \to 0 \) when \( t \to \infty \). Obviously \( y_1 \to 0 \) when \( t \to \infty \).

The above results can be summarized in the following theorem.

**Theorem 1:** For the chained multiple mass-spring-damper system (6) and the re-constructed dynamic model represented by equations (15) to (17), the regulation of the position \( y_1 \) is achieved (\( y_1 \to 0 \) when \( t \to \infty \)) under the control law (28) and the parameter adaptation laws (24) and (33).

**Remark 3:** The CMMSD system considered is assumed to be free of external disturbances. To keep the robustness of the controlled system under the external disturbances, various robustification approaches can be used, such as dead-zone modification or \( \delta \)-modification \([10,14]\), though the resulting controllers tend to be more complicated. As pointed out in \([10]\) and \([11]\), the adaptive controller developed with backstepping methods shows much higher degree of robustness than that of conventional adaptive controller even in the absence of robustification tools.

3.2. Model Reference Adaptive Output Feedback Control

In this section, adaptive output feedback control is to be designed based on MRAC approach. To begin with, the system transfer function (5) is re-written using the differential operator \( p = d/dt \) such that

\[
y_1 = \frac{d^{2n}u}{A_p}, \tag{34}
\]

where

\[
A_p = p^{2n} + \sum_{j=0}^{2n-1} a_{2n,j} p^j.
\]

Assume that the desired behavior of the controlled system is specified by the following reference model:

\[
y_{1m} = \frac{B_m}{A_m} u_c, \tag{35}
\]

where \( B_m = d_m \), \( A_m = p^{2n} + \sum_{j=0}^{2n-1} a_{m,j} p^j \), \( u_c \) is the command input and \( y_{1m} \) is the desired output. Note that \( A_m \) is a stable monic polynomial with the same order as that of \( A_p \) and it should be selected such that when \( u_c = 0 \), \( y_{1m} \to 0 \).

The task now is to find a control input \( u \) for system (34) such that the controlled system follows the reference model (35). Following the pole placements procedure in \([15]\), this objective can be achieved by making the control \( u \) to meet the equations:

\[
\begin{align*}
P_r u &= P_m u_c - P_s y_1, \\
P_s &= s_{2n-1} p^{2n-1} + s_{2n-2} p^{2n-2} + \ldots + s_1 p + s_0, \\
P_i &= t_{2n-1} p^{2n-1} + t_{2n-2} p^{2n-2} + \ldots + t_1 p + t_0, \\
P_r &= p^{2n-1} + r_{2n-2} p^{2n-2} + \ldots + r_0, \\
P_i &= P_o B_m/d_{2n}
\end{align*}
\]

where \( P_s \) and \( P_i \) are polynomials and their coefficients \( s_j, t_i \) and \( r_j \) \((i = 0, 1, \ldots, 2n-1)\) can be obtained from the following equations:

\[
\begin{align*}
A_p P_r + d_{2n} P_i &= P_o A_m, \tag{37} \\
P_i &= P_o B_m/d_{2n}
\end{align*}
\]

with \( P_o \) being a pre-defined observer polynomial \( P_o = p^{2n-1} + a_{2n-2} p^{2n-2} + \ldots + a_n + a_0 \) with \( a_i \) being its coefficients. Equation (37) is normally called Diophantine equation.

From equations (37) and (38), the coefficients of \( P_r \), \( P_s \) and \( P_i \) are obtained:

\[
\begin{align*}
r_4 &= a_{2n-1} - a_{2n-2}, \\
&= a_{2n-1} + \sum_{j=0}^{2n-2} (o_j a_{m,2n+j-i} - n_j a_{2n+j-i}) \quad (i = 0, \ldots, 2n-2), \\
s_i &= \sum_{j=0}^{2n-2} (a_m a_{m,i-j} - n_j a_{i-j})/d_{2n} \quad (i = 0, \ldots, 2n-2), \\
t_i &= d_m a_{i}/d_{2n} \quad (i = 0, \ldots, 2n-2).
\end{align*}
\]

Obviously \( r_i, s_i \) and \( t_i \) are the functions of \( a_{2n,i}, a_{m,i}, o_i \) and \( d_{2n} \). In the following, we will develop the parameter adaptation laws to estimate these parameters.

From equations (34), (35) and (36), the error between the output of the controlled loop and the reference model is obtained:

\[
ed = y_1 - y_{1m} = \frac{d^{2n}}{P_o A_m} (P_r u + P_s y_1 - P_m u_c). \tag{39}
\]

To express this error in a linear-in-parameter (LIP) form, re-arrange equation (39) such that

\[
ed = d_{2n} \left( \frac{1}{P_r} P_r - P_s + \frac{P_s}{P_f} P_r - \frac{P_s}{P_f} y_1 - \frac{P_f}{P_f} u_c \right),
\]

where \( P_r = P_1 P_2 \), \( P_s = A_m \) and \( P_1 = P_2 = P_0 \). Obviously \( P_r - P_2 \) is a polynomial of \( p \) with coefficients being \( n_i = r_i - a_i \) \((i = 0, 1, \ldots, 2n-1)\).

Define a vector consisting of coefficients of the polynomials of \( P_r - P_2 \), \( P_s \) and \( P_i \) such that

\[
\theta_f = [r_{2n-2} \ldots s_{2n-1} \ldots s_0 t_{2n-1} \ldots t_0]'
\]

and another vector consisting of filtered input, output and the command inputs such that

\[
\begin{align*}
P_r u &= P_m u_c - P_s y_1, \\
P_s &= s_{2n-1} p^{2n-1} + s_{2n-2} p^{2n-2} + \ldots + s_1 p + s_0, \\
P_r &= p^{2n-1} + r_{2n-2} p^{2n-2} + \ldots + r_0, \\
P_i &= P_o B_m/d_{2n}
\end{align*}
\]
With $\varphi$ and $\theta_f$ defined above, the error $e$ is expressed as

$$e = d_{2n}\eta,$$

$$\varepsilon = d_{2n}\varphi^T \hat{\theta}_f,$$

$$\eta = \frac{1}{P_1} u - \varphi^T \hat{\theta}_f,$$  \hspace{1cm} (42)

where $\hat{\theta}_f = \theta_f - \tilde{\theta}_f$ and $\tilde{\theta}_f$ is the estimate of $\theta_f$. It is obvious that $\varepsilon$ is linear in the parameters. The adaptation law for $\theta$ is then obtained through gradient approach [15]:

$$\dot{\hat{\theta}}_f = \gamma_f \varphi \varepsilon,$$

where $\gamma_f > 0$ is the adaptation gain. Note that the unknown parameter $d_{2n}$ is absorbed in $\gamma_f$. The calculation of $\zeta$ requires unknown parameter $d_{2n}$. To estimate it as well as $\theta_f$, the augmented error $e$ is replaced by prediction error

$$e_p = e - \dot{d}_{2n}(\varphi^T \hat{\theta}_f + \frac{1}{P_1} u)$$

and the estimates of $\theta_f$ and $d_{2n}$ are thus given by

$$\dot{\hat{\theta}}_f = \gamma_f \varphi e_p,$$  \hspace{1cm} (43)

$$\dot{\hat{d}}_{2n} = \gamma_f (\varphi^T \hat{\theta}_f + \frac{1}{P_1} u)e_p.$$  \hspace{1cm} (44)

Given $u_c = 0$ and following the same lengthy procedure in proving the stability of general MRAC controllers [15,16], it can be showed that under the controller (41) and parameter adaptation laws (43) and (44), $y_1 \to y_{1m}$ asymptotically and the signals in the controlled system are all bounded. As $y_{1m} \to 0$ when $u_c = 0$, thus we can conclude $y_1 \to 0$ asymptotically. The regulation of the output $y_1$ with $u$ is thus achieved. The above results are summarized in the following theorem.

**Theorem 2:** For the chained mass-spring-damper system with transfer function (34), the position $y_1$ is controlled such that $y_1 \to 0$ when $t \to \infty$ under the control law (41) and the parameter adaptation laws (43) and (44).

**Remark 4:** For adaptive feedback control with backstepping, MRAC based approach needs more control parameters such as those of the reference model, observer polynomials, filters and the adaptation gains. Several filters are also needed to filter the outputs, command inputs and the control inputs respectively.
Remark 5: Observer polynomial $P_o$ should be selected such that it is stable with faster dynamic response than that of $A_m$.

4. SIMULATION STUDY

Consider a system with three mass-spring-damper units. The system parameters are selected as $m_1 = m_2 = m_3 = 2\text{kg}$, $b_1 = b_2 = b_3 = 0.8\text{N/ms}^{-1}$ and $k_1 = k_2 = 40\text{N/m}$. Based on these system parameters, it can be calculated that $\theta=[200 1.0 80 64 1212 480 0]^T$ and $d=0.005$. As $\theta$ and $d$ are unknown, their initial estimates are assumed to be 20% of their true values, that is, $\hat{\theta} = [40 0.2 16 13 240 96 0]^T$ and $\hat{d} = 0.0015$ respectively. Assume initially $y_1 = q_1 = 0.02$ (0.02m displacement from its equilibrium position) and $q_2 = q_3 = \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$.

First, the simulation is done for the adaptive backstepping output feedback controller. The input to the system is obtained by setting $n=3$ and setting control parameters $c_1 = 2$, $c_2 = 3$, $c_3 = 5$, $c_4 = 4$, $c_5 = 6$, $c_6 = 7$, $\lambda_1 = 10$, $\lambda_2 = 50$, $\lambda_3 = 20$, $\lambda_4 = 30$, $\lambda_5 = 20$, $\lambda_6 = 20$, $\Gamma = 25\text{sec}^6$ and $\gamma = 20$. The system responses are shown from Figs. 2 to 4. The control $u$ is shown in Fig. 5. The estimates for $\hat{\theta}$ are shown in Figs. 3 and 4. $\hat{d}$ is plotted together with its value under disturbances in Fig. 10.

It can be seen that positions $q_i$ ($i=1,2,3$) approach to zero in approximately 4 seconds. It is interesting to note that, during the transient period, $q_3$ demonstrates larger oscillations than those of $q_2$ and $q_1$ while that of $q_1$ is the minimum. It is understandable as $q_3$ is directly affected by the input $u$. The control input is in a reasonable range, though it shows a sharp change to overcome the initial position error at the beginning. Under the adaptation laws, all the parameter estimates become stable around 2.5 seconds, though they do not approximate their true values.

To test the robustness of the controller under the bounded external disturbances, a small bounded disturbance $\Delta(t) = [\delta_1(t) \delta_2(t) \ldots \delta_6(t)]^T$ is added to the system dynamics such that

$$\dot{x} = A_x x + b_x u + \Delta(t),$$

$$x_1 = c_1^T x,$$

where $\delta_j(i=12\ldots 6)$ is defined as

$$\delta_j = \begin{cases} 0.15 & t \leq 2.5 \text{Sec} \\ 0.1 & t > 2.5 \text{Sec}. \end{cases}$$

The simulation results are plotted from Figs. 6 to 8 and Fig. 10. The control effort is plotted in Fig. 9. It can be observed that frequent oscillations are superimposed with the position signals for a longer time compared with those when the system is external disturbance free. Though the asymptotic regulation of the positions are not achieved, the position errors tend to be bounded within a narrow bound. The parameter estimates are stabilized. Though it is observed that the output of the system becomes divergent if the disturbances is large, the above simulation results still show that the controller demonstrates has some degrees of robustness to some bounded disturbances. For larger disturbances, robust control approach should be used for compensations.

The simulation is also done for MRAC adaptive output feedback controller. The initial states of the sys-
Fig. 6. Position responses under disturbances
(Solid: $q_1$, dashed: $q_2$, dashdot: $q_3$).

Fig. 7. Parameter estimates under disturbances-Part 1
(Solid: $\hat{\theta}_1$, dashed: $\hat{\theta}_2$, dotted: $\hat{\theta}_3$, dashdot: $\hat{\theta}_4$).

Fig. 8. Parameter estimates under disturbances-Part 2
(Solid: $\hat{\theta}_5$, dashed: $\hat{\theta}_6$, dotted: $\hat{\theta}_7$).

Fig. 9. Control input under disturbances $u$.

Fig. 10. Parameter estimates $\hat{d}$ (Solid: $\hat{d}$ when free of disturbances, dashed: $\hat{d}$ under disturbances).

Fig. 11. Position response under the MRAC adaptive output feedback control
(Solid: $q_1$, dashed: $q_2$, dashdot: $q_3$).
system are the same as those for backstepping controller. The controller parameters of the controller are chosen as $\gamma = 12$. The initial values of the parameter estimates $\hat{\theta}_1$ and $\hat{d}_2$ are set to zeros. The simulation results are shown from Fig. 11 to Fig. 13. It can be seen that it takes longer time (almost 50 seconds) for the output to be regulated to zero compared with that of backstepping control scheme. The position responses also show bigger oscillations (especially $q_3$) in a longer period. The parameter estimates are convergent, though they are not close to their true values. The control input is in a reasonable range with a magnitude much smaller than that of backstepping controller, but its oscillation lasts longer. In the simulation, it is found that the controller is sensitive to the controller parameters and the initial values of the estimated parameters. The outputs becomes divergent once the disturbance $\Delta(t)$ is added to the system.

5. CONCLUSION

In this paper, dynamic models and adaptive output feedback controller for position regulation are developed for general chained multiple mass spring damper systems. Based on backstepping and MRAC adaptive methods, two adaptive output feedback controllers—recursive and non-recursive respectively, are developed. They rely on the input and the output of the system and do not require the exact knowledge of the parameters and the internal states of the system. Under the proposed controllers, the output (position) of the system is regulated to zero. The simulations are used to verify and compare the effectiveness of the proposed control approaches.

APPENDIX 1: TRANSFER FUNCTION $H_{2n}(s)$

The transfer function $H_{2n}(s)$ can be derived step wise from

$$H_{2i}(s) = \frac{d_{2i}}{s^2 + \sum_{j=0}^{2i-1} a_{2i,j}s^j} (i = 1 \ldots n), \quad (45)$$

where

$$d_{2i} = m_{i-1}d_{2i-2},$$

$$a_{2i,j} = a_{2i-2,j-2} + m_{i-1}h_{j}(a_{2i-2,j-1} + \sigma(j-2i+1))$$

$$+ m_{i-1}k_{j-1}(a_{2i-2,j} + \sigma(j-2i+2))$$

$$- d_{2i-4}a_{2i-2}m_{i-1}k_{j-1}(a_{2i-4,j} + \sigma(j-2i+4)),$$

$$a_{2i,j} = a_{2i-2,j-2} + m_{i-1}h_{j}(a_{2i-2,j-1} + \sigma(j-2i+1))$$

$$+ m_{i-1}(k_{j-1} + k_{j})(a_{2i-2,j} + \sigma(j-2i+2))$$

$$- d_{2i-4}a_{2i-2}m_{i-1}k_{j-1}(a_{2i-4,j} + \sigma(j-2i+4)),$$

$$l = 2, 3, \ldots i - 1, \quad j = 0, 1, 2, \ldots l - 1,$$

with function $\sigma(\bullet)$ defined as

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \neq 0. \end{cases} \quad (46)$$

This closed-form solution for the coefficients in the system transfer function (45) is very convenient for the design of the controllers, adaptive or nonadaptive, model free or model based. For this reason, we would like to share it with control communities.

REFERENCES


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